M E T U
Department of Mathematics

<table>
<tr><td colspan="2">Introduction to Differential Equations
Final</td></tr>
<tr><td>Code : Math 219
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Time : 13:30
Duration : 120 minutes</td><td>Last Name :
Name :   Student No. :
Department :   Section :
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4 QUESTIONS ON 4 PAGES
TOTAL 100 POINTS</td></tr>
</table>

1	2	3	4		SHOW YOUR WORK	

Question 1 (25 pts) Find the solution of the IVP

$$y'' + 2y' + 2y = \cos t - \delta\left(t - \frac{\pi}{2}\right), \quad y(0) = 0, y'(0) = 0.$$

Take Laplace transform of both sides of the eqn.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t - \delta(t - \tfrac{\pi}{2})\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\} - \mathcal{L}\{\delta(t - \tfrac{\pi}{2})\} \qquad \text{let } \mathcal{L}\{y(t)\} = Y(s)$$

$$s^2 Y(s) - s\overset{0}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + 2sY(s) - 2\overset{0}{\cancel{y(0)}} + 2Y(s) = \frac{s}{s^2+1} - e^{-\frac{\pi}{2}s}$$

$$\Rightarrow \quad Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} - e^{-\pi/2\, s} \cdot \frac{1}{s^2+2s+2}$$

$$\text{So} \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+2)}\right\} - \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s} \cdot \frac{1}{(s+1)^2+1}\right\}$$

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1}_{(s^2+2s+2)} + \frac{Cs+D}{s^2+2s+2}_{(s^2+1)} \iff As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Ds^2 + Cs + D = s$$

$$\iff \left.\begin{array}{ll} A + C = 0 & \Rightarrow C = -A \\ 2B + D = 0 & \Rightarrow D = -2B \\ B + 2A + D = 0 & \\ 2B + 2A + C = 1 & \end{array}\right\} \begin{array}{l} 2B + A = 1 \\ 2A - B = 0 \\ \iff A = \frac{1}{5} \quad B = \frac{2}{5} \end{array}$$

$$C = -\frac{1}{5} \quad D = -\frac{4}{5}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{s}{s^2+1} + \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{(s+1)^2+1} - \frac{4}{5}\frac{1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\underbrace{\frac{1}{(s+1)^2+1}}_{F(s)}\right\}$$

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{4}{5}e^{-t}\sin t - u_{\frac{\pi}{2}}(t) \cdot \underbrace{e^{-(t-\frac{\pi}{2})} \cdot \sin\left(t - \frac{\pi}{2}\right)}_{f(t-\frac{\pi}{2})}$$

Question 2 (25 pts) Find all solutions of the system

$$\mathbf{x}' = \overbrace{\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}}^{A} \mathbf{x} + \overbrace{\begin{bmatrix} t^2 \\ t^{-2} \end{bmatrix}}^{g(t)}, \quad t > 0.$$

First, find corresponding homogeneous equations solution, i.e. solve $\vec{X}' = A\vec{X}$

$$\Rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(-2-\lambda)+4 = 0 \Leftrightarrow \lambda^2 = 0 \Leftrightarrow \lambda = 0 \text{ (repeated twice)}$$

$\underline{\lambda = 0}$ for corresponding eigenvector; $A.\vec{P_1} = \vec{0}$;

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow 2\xi_1 + 4\xi_2 = 0 \Rightarrow \begin{bmatrix} -2\xi_2 \\ \xi_2 \end{bmatrix} \quad \text{let } \xi_2 = 1 \text{ choose } \vec{P_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for generalized eigenvector $\quad A.\vec{P_2} = 0.\vec{P_2} + \vec{P_1}$ i.e. $A.\vec{P_2} = \vec{P_1} \Rightarrow$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & 4 & -2 \\ -1 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & 4 & -2 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 2\xi_2 - 1 \\ \xi_2 \end{bmatrix}$$

let $\xi_2 = 0$ so choose $\vec{P_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ So $\vec{X_1}(t) = \vec{P_1}.e^{\lambda t} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

& $\vec{X_2}(t) = \vec{P_1}te^{0.t} + \vec{P_2}e^{0.t} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}.1 = \begin{bmatrix} -2t-1 \\ t \end{bmatrix}$

$\vec{X_h}(t) = c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2t-1 \\ t \end{bmatrix}$ + A fundamental matrix is $\Psi(t) = \begin{bmatrix} -2 & -2t-1 \\ 1 & t \end{bmatrix}$ since

$\det(\Psi(t)) = -\cancel{2t} + \cancel{2t} + 1 = 1 > 0$

For particular solution; $\vec{X_p} = \Psi(t).\vec{u}(t)$ where $\Psi(t).\vec{u}'(t) = \begin{bmatrix} t^2 \\ t^{-2} \end{bmatrix}$

$$\Rightarrow u_1'(t) = \frac{\begin{vmatrix} t^2 & -2t-1 \\ t^{-2} & t \end{vmatrix}}{\det(\Psi(t))} = \frac{t^3 + \frac{2}{t} + \frac{1}{t^2}}{1} \Rightarrow u_1(t) = \int\left(t^3 + \frac{2}{t} + \frac{1}{t^2}\right)dt = \frac{t^4}{4} + 2\ln t - \frac{1}{t}$$

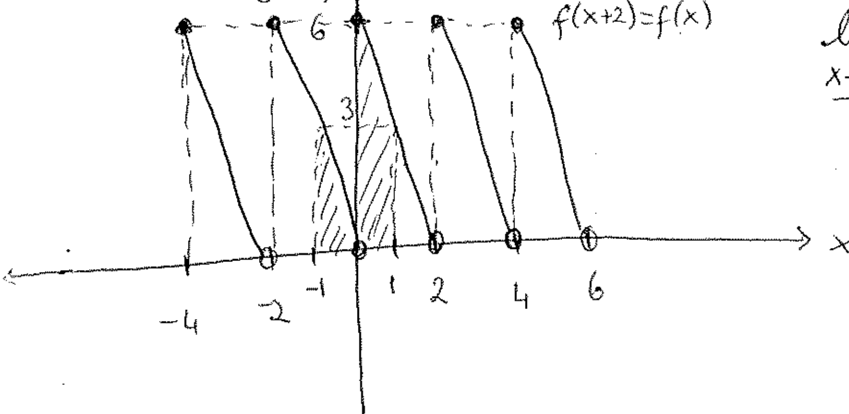
$$u_2'(t) = \frac{\begin{vmatrix} -2 & t^2 \\ 1 & t^{-2} \end{vmatrix}}{\det(\Psi(t))} = \frac{-\frac{2}{t^2} - t^2}{1} \Rightarrow u_2(t) = \int\left(-\frac{2}{t^2} - t^2\right)dt = \frac{2}{t} - \frac{t^3}{3}$$

$$\therefore \vec{X_p} = \begin{bmatrix} -2 & -2t-1 \\ 1 & t \end{bmatrix}\begin{bmatrix} \frac{t^4}{4} + 2\ln t - \frac{1}{t} \\ \frac{2}{t} - \frac{t^3}{3} \end{bmatrix} = \begin{bmatrix} -\frac{t^4}{2} - 4\ln t + \cancel{\frac{2}{t}} - 4 + \frac{2}{3}t^4 - \cancel{\frac{2}{t}} + \frac{t^3}{3} \\ \frac{t^4}{4} + 2\ln t - \frac{1}{t} + 2 - \frac{t^4}{3} \end{bmatrix}$$

$$\vec{X_g} = \vec{X_h} + \vec{X_p} = c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2t-1 \\ t \end{bmatrix} + \begin{bmatrix} \frac{t^4}{6} - 4\ln t + \frac{t^3}{3} \\ -\frac{t^4}{12} - \frac{1}{t} + 2\ln t + 2 \end{bmatrix}$$

Question 3 (25 pts) Let $f(x) = 6 - 3x$ for $0 \le x < 2$ and suppose that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

(a) Draw a graph of the function $f(x)$. To which value does the Fourier series of $f(x)$ converge to, at $x = 0$?



$$\frac{\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)}{2} = \frac{f(0^+)+f(0^-)}{2} = \frac{6+0}{2} = 3$$

So Fourier series of $f(x)$ converge to 3 at $x=0$.

(b) Find the Fourier series representation of $f(x)$.

By using Euler-Fourier Formulas, since f is periodic with period $2L = 2$ & f and f' are precewise cont. on $-1 \le x < 1$

then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} -3x\,dx + \int_{0}^{1}(6-3x)\,dx = -\frac{3}{2}x^2\Big|_{-1}^{0} + 6x - \frac{3}{2}x^2\Big|_{0}^{1} = 6$

$\Rightarrow a_0 = 6 \Rightarrow \frac{a_0}{2} = 3$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx = 1\int_{-1}^{0}(-3x)\cos\frac{n\pi x}{1}\,dx + \int_{0}^{1}(6-3x)\cos\left(\frac{n\pi}{1}x\right)dx$

$\Rightarrow a_n = -\int_{-1}^{0} 3x\cos(n\pi x)\,dx + \int_{0}^{1}(6-3x)\cos(n\pi x)\,dx = -3\int_{-1}^{0} x\cos(n\pi x)\,dx - 3\int_{0}^{1} x\cos(n\pi x)\,dx + 6\int_{0}^{1}\cos(n\pi x)\,dx$

$= \underbrace{-3\int_{-1}^{1} \underbrace{\underbrace{x}_{odd}\underbrace{\cos(n\pi x)}_{even}}_{odd}\,dx}_{0} + \underbrace{6\frac{1}{n\pi}\sin n\pi x\Big|_{0}^{1}}_{0} = 0$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 1\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}(-3x)\sin(n\pi x)\,dx + \int_{0}^{1}(6-3x)\sin(n\pi x)\,dx$

$\Rightarrow b_n = -3\int_{-1}^{0} x\sin(n\pi x)\,dx - 3\int_{0}^{1} x\sin(n\pi x)\,dx + 6\int_{0}^{1}\sin(n\pi x)\,dx = -3\int_{-1}^{1} x\sin(n\pi x)\,dx + 6\int_{0}^{1}\sin n\pi x\,dx$

$= 3\left[x\frac{1}{n\pi}\cos(n\pi x) - \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_{-1}^{1}\right] + 6\left(\frac{1}{n\pi}\cos(n\pi x)\Big|_{0}^{1}\right) = \frac{6}{n\pi}\cos 0 = \frac{6}{n\pi}$

$\therefore f(x) = 3 + \sum_{n=1}^{\infty}\left(0 \cdot \cos\frac{n\pi x}{1} + \frac{6}{n\pi}\sin\frac{n\pi x}{1}\right) = 3 + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)$

Question 4 (25 pts) Consider the heat conduction problem

$$(*) \quad 2u_{xx} = u_t, \qquad 0 < x < 6, \; t > 0$$
$$(**) \qquad u_x(0,t) = 0, \quad u_x(6,t) = 0$$

(a) By using separation of variables, show that the problem has non-trivial solutions of the form $u_n(x,t) = \cos\left(\frac{n\pi x}{6}\right) \cdot T_n(t)$ for each nonnegative integer n.

Let $u(x,t) = X(x).T(t)$ then $u_x = X'T$, $u_{xx} = X''T$ & $u_t = XT'$ put into $(*)$

so $2X''T = XT' \Rightarrow \frac{X''}{X} = \frac{1}{2}\frac{T'}{T} = -\lambda \Rightarrow$ $X'' + \lambda X = 0$ ①, $\frac{T'}{T} = -2\lambda$ ②

From $(**)$ $\left.\begin{array}{l} u_x(0,t) = X'(0).T(t) = 0 \\ u_x(6,t) = X'(6).T(t) = 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} X'(0) = X'(6) = 0 \\ \text{or } T(t) = 0 \Rightarrow u(x,t) = 0 \end{array}\right\}$ So we get

$\boxed{\cosh\mu x = \frac{e^{\mu x}+e^{-\mu x}}{2} \quad \sinh\mu x = \frac{e^{\mu x}-e^{-\mu x}}{2}}$

a B.V.P $X'' + \lambda X = 0$ & $X'(0) = X'(6) = 0$

Case 1 – $\lambda < 0$ i.e. $\lambda = -\mu^2 \Rightarrow X'' - \mu^2 X = 0 \Rightarrow X(x) = c_1 \sinh\mu x + c_2\cosh\mu x$, $\mu \neq 0$

$X'(x) = \mu c_1 \cosh\mu x + c_2\mu\sinh\mu x$ $\quad X'(0) = \mu c_1 . 1 + c_2\mu . 0 = 0 \Leftrightarrow c_1 = 0$

$X'(6) = c_1\mu\cosh 6\mu + c_2\mu\sinh 6\mu = 0 \Leftrightarrow c_2 = 0$ } no non-trivial soln.

Case 2. $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 x + c_2$ $\quad X'(x) = c_1 = 0$ $\quad x = 0, 6 \Rightarrow c_1 = 0$

so $\lambda = 0$ is an eigenvalue & $X(x) = 1$ is an eigenfunction

Case 3 – $\lambda > 0$ i.e. $\lambda = \mu^2 \Rightarrow X'' + \mu^2 X = 0 \Rightarrow X(x) = c_1\cos\mu x + c_2\sin\mu x$ $\quad X'(6) = -\mu c_1 \sin 6\mu = 0$

$X'(x) = -\mu c_1 \sin\mu x + \mu c_2\cos\mu x$ $\quad X'(0) = -\mu c_1 \sin 0 + \mu c_2 \cos 0 = 0 \Rightarrow c_2 = 0$

$\Leftrightarrow 6\mu = n\pi \Leftrightarrow \mu = \frac{n\pi}{6} \Rightarrow \lambda_n = \frac{(n\pi)^2}{36}$ are eigenvalues $n = 1, 2, 3, \ldots$ $X_n = \cos\frac{n\pi}{6}x$; $n = 1, 2, \ldots$ are eigenfunctions.

So, ② is $T_n(t) = e^{-2\frac{(n\pi)^2}{36}t}$ so $u_n(x,t) = e^{-\frac{2(n\pi)^2}{36}t} . \cos\frac{n\pi}{6}x$

(b) Find the solution of the problem subject to the initial condition

$$u(x,0) = \pi + 3\cos(\pi x) - 4\cos\left(\frac{3\pi x}{2}\right) - \cos(3\pi x).$$

$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\frac{2(n\pi)^2}{36}t} . \cos\left(\frac{n\pi}{6}x\right)$ $\quad \frac{n\pi}{6}x \Rightarrow n = 6$ $\quad \frac{n\pi}{6} = \frac{3\pi}{2} \Rightarrow n = 9$ $\quad \frac{n\pi}{6} = 3\pi \Rightarrow n = 18$

$u(x,0) = \sum_{n=0}^{\infty} c_n . 1 . \cos\left(\frac{n\pi}{6}x\right) = \underbrace{\pi}_{c_0} + \underbrace{3}_{c_6}\cos(\pi x) - 4\cos\left(\frac{3\pi}{2}x\right) - \cos(3\pi x)$

$\therefore$ $c_0 = \pi$ $\quad c_6 = 3$ $\quad c_9 = -4$ $\quad c_{18} = -1$ $\quad c_i = 0 \quad \forall i \neq 0, 6, 9, 18$

Hence $u(x,t) = \pi + 3e^{-2\pi^2 t}\cos(\pi x) - 4e^{-\frac{9\pi^2}{2}t}\cos\left(\frac{3\pi}{2}x\right) - e^{-18\pi^2 t}\cos(3\pi x)$

(c) What is the steady state solution? (In other words, what is $\lim_{t\to\infty} u(x,t)$?)

For steady state solution:

$$\lim_{t\to\infty}\left[\pi + 3\underbrace{e^{-2\pi^2 t}}_{\to 0}\cos(\pi x) - 4\underbrace{e^{-\frac{9\pi^2 t}{2}}}_{\to 0}\cos\left(\frac{3\pi}{2}x\right) - \underbrace{e^{-18\pi^2 t}}_{\to 0}\cos(3\pi x)\right] = \pi \quad //$$